

Wavelet Analysis of Inhomogeneous Data with application to the Cosmic Velocity Field

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Abstract. In this article we give an account of a method of smoothing spatial inhomogeneous data sets by using wavelet reconstruction on a regular grid in an auxiliary space onto which the original data is mapped. In a previous paper by the present authors, we devised a method for inferring the velocity potential from the radial component of the cosmic velocity field assuming an ideal sampling. Unfortunately the sparseness of the real data as well as errors of measurement require us to first smooth the velocity field as observed on a 3-dimensional support (i.e. the galaxy positions) inhomogeneously distributed throughout the sampled volume. The wavelet formalism permits us to introduce a minimal smoothing procedure that is characterized by the variation in size of the smoothing window function. Moreover the output smoothed radial velocity field can be shown to correspond to a well defined theoretical quantity as long as the spatial sampling support satisfies certain criteria. We argue also that one should be very cautious when comparing the velocity potential derived from such a smoothed radial component of the velocity field with related quantities derived from other studies (e.g : of the density field).

1. Introduction

An ‘inverse’ problem posed frequently in cosmology is to construct the velocity potential (assuming an irrotational velocity field) from observations of only the ‘radial’ component of the velocity field. We have argued elsewhere [7] that wavelet transforms provide a convenient formalism for this procedure, but subsequently in the course of our practical investigations we found that the signal is easily distorted by that due to the inhomogeneity of a non-uniformly sampled data set. Such a sample is constituted by a set of galaxy peculiar velocities distributed (as they in practice always are) over a nearly random support, and it is to this example that we address most of our remarks in this paper. However the solution that we propose, namely a non-linear algorithmic mapping from the real space into a fictitious but uniformly sampled space may be of general interest and is therefore reported here.

The ultimate cosmological objective is to extract information concerning the fluctuations of the total mass in the universe $\delta_t(\mathbf{x})$ from the observed radial component $v_r(\mathbf{x})$ of the cosmic peculiar velocity field $\mathbf{v}(\mathbf{x})$, assuming that the latter has been created by fluctuations in the self-gravitating matter. One of the goals is indeed, by comparing the fluctuations of the total mass obtained as above with the fluctuations of the luminous mass $\delta_l(\mathbf{x})$ obtained from the study of the distribution of the galaxies in the universe, to characterize the distribution of the dark matter in the universe $\delta_d(\mathbf{x}) = \delta_t(\mathbf{x}) - \delta_l(\mathbf{x})$.

According to the standard models of large scale structure formation, there are good reasons to believe that the cosmic velocity field is irrotational at least above some ill-defined scale s_c . Bertschinger and Dekel [1] have pointed out that consequently above the scale s_c , the cosmic velocity field $\mathbf{v}(\mathbf{x})$ may be derived from a potential ($\mathbf{v}(\mathbf{x}) = \nabla \Phi(\mathbf{x})$, i.e : \mathbf{v} is curl-free), and that this kinematical potential Φ can be extracted by integrating the observed radial component of the velocity field $v_r(\mathbf{x})$ along the line-of-sight :

$$\Phi(\mathbf{x}) = (Pv_r)(\mathbf{x}) = \int_0^1 dl v_r(l\mathbf{x}) \quad (1.1)$$

Moreover, above the scale s_c , the velocity fluctuations still belong to the linear regime and the total mass density field $\delta_t(\mathbf{x})$ is thus linked to the kinematical potential $\Phi(\mathbf{x})$ through the Poisson equation ($\delta_t(\mathbf{x}) \propto \nabla^2 \Phi(\mathbf{x})$).

In our previous paper [7], we devised a method based on the properties of the wavelet transform for inferring the kinematical potential from the observed radial component of the cosmic velocity field. We summarize in section 2 why our method is well adapted for

solving the problem of the choice of the scale s_c , and how we can directly derive from the radial velocity field quantities such as the laplacian of the kinematical potential (and thus the fluctuation in the total mass).

Unfortunately, as was observed above, the radial velocity field is measured with large statistical uncertainties and is sampled on a support (i.e : the positions of galaxies) inhomogeneously distributed throughout space. Thus one has to first smooth the observed radial velocity field before applying no matter which inversion procedure to find the kinematic potential and hence the reconstructed complete velocity field. The POTENT inversion [2,4] has already given impressive results. However, we see two limitations in the smoothing scheme used in the POTENT procedure. Firstly, the size of their smoothing window function doesn't vary spatially, and so fictitious information is added in undersampled regions and a significant part of the signal is lost in oversampled regions. Secondly, the errors intervening during the smoothing procedure of the POTENT method are difficult to quantify. In this article, in section 3, we present a smoothing procedure based on the wavelet analysis, which permits one to *optimally* smooth a field sampled on a 3-dimensional support distributed inhomogeneously throughout space. The natural variation in size of the smoothing window function in our procedure permits us to realize a minimal or optimal smoothing procedure (i.e : without loss of information). Moreover we prove that our smoothed output field corresponds to a theoretical quantity (defined by the wavelet analysis formalism), so permitting us to quantify and control the errors intervening during the successive steps of our smoothing procedure.

Finally, we analyze in a general way in section 4 the wavelet transform procedure involved in the reconstruction of the kinematical potential given the smoothed radial component of the velocity field. We show that this operation of reconstruction doesn't commute with the preliminary smoothing operation. This creates difficulties already at the 'a priori' theoretical level when attempting to compare the reconstructed velocity field or its associated mass density field with their correspondant quantities derived from other studies, since in those cases the smoothing may be effected in a different order. This problem is general in the sense that it is in no way unique to our smoothing procedure but rather depends on the global character of equation (1.1).

2. Reconstruction of the velocity field using wavelet analysis

2.1. The decomposition by scales

One of the characteristic features of wavelet analysis is that it allows a simultaneous study of both the positional and the scaling properties of a function. Thanks to the

wavelet reconstruction theorem (see [6]), a square integrable function (for example the radial velocity field $v_r(\mathbf{x})$ defined at each point \mathbf{x} of our 3-dimensional space) can be decomposed in a family of functions $v_r^{(s)}(\mathbf{x})$ as follows :

$$v_r(\mathbf{x}) = (Wv_r)(\mathbf{x}) = \int_0^\infty \frac{ds}{s} v_r^{(s)}(\mathbf{x}) \quad (2.1)$$

The integral is performed over all scales s and $v_r^{(s)}(\mathbf{x})$ is the component of $v_r(\mathbf{x})$ smoothed on the scale s . It is in fact the spatial convolution of the radial velocity field $v_r(\mathbf{x})$ with the "reproducing kernel" $K(s, \mathbf{x}, \mathbf{y})$ (see the illustration figure 1a), a well-defined function centered on \mathbf{x} and of spatial extension s (see [7]) namely :

$$v_r^{(s)}(\mathbf{x}) = \int_{-\infty}^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty d^3\mathbf{y} K(s, \mathbf{x}, \mathbf{y}) v_r(\mathbf{y}) \quad (2.2)$$

$v_r^{(s)}(\mathbf{x})$ thus contains information localized "around" the scale s in frequency (scale) space. As the scale s decreases, a more and more detailed picture of the velocity field is thus available. Because of the redundant properties of the continuous wavelet transform, the integral involved in equation 2.1 can in practice be replaced by a discrete sum over a few scales (see [3,7]). Moreover, the reconstruction will necessarily be limited to a maximum resolution (minimum scale s_c) by the finite spatial resolution of the data distribution.

To emphasize this latter point let us introduce two operators W_{s_c} and W^{s_c} acting on v_r as follows :

$$v_r = Wv_r = W_{s_c}v_r + W^{s_c}v_r \quad (2.3)$$

$$(W_{s_c}v_r)(\mathbf{x}) = \int_{s_c}^\infty \frac{ds}{s} v_r^{(s)}(\mathbf{x}) \quad (2.4)$$

$$(W^{s_c}v_r)(\mathbf{x}) = \int_0^{s_c} \frac{ds}{s} v_r^{(s)}(\mathbf{x}) \quad (2.5)$$

$(W^{s_c}v_r)(\mathbf{x})$ contains the information about $v_r(\mathbf{x})$ at scales smaller than s_c and the term $(W_{s_c}v_r)(\mathbf{x})$ may be considered to be a smoothed version of the radial velocity field determined by a smoothing window function $g(s_c, \mathbf{x}, \mathbf{y})$ of size s_c namely :

$$g(s_c, \mathbf{x}, \mathbf{y}) = \int_{s_c}^\infty \frac{ds}{s} K(s, \mathbf{x}, \mathbf{y}) \quad (2.6)$$

Note then how in practise one performs the reconstruction of v_r . The $v_r^{(s)}(\mathbf{x})$ components involved in the integral over the scales in equation (2.3) are summed step by step, proceeding from the larger cutoff scales to the smaller ones. When the desired level of spatial resolution is reached (for example at a given scale S), we then halt the reconstruction and so obtain the smoothed version of $v_r(\mathbf{x})$ at this scale S , $(W_Sv_r)(\mathbf{x})$. This procedure

Figure 1: For a given scale s and position \mathbf{x} , the variations on a 2-dimensional cut of : (a) the "reproducing kernel" $K(s, \mathbf{x}, \mathbf{y})$, (b) the "generalized kernel" $L(s, \mathbf{x}, \mathbf{y})$.

is particularly convenient in the case of our study for which the radial velocity field has to be first smoothed at an ill-defined scale s_c in order to eliminate the small-scale rotational component of the cosmic velocity field. Our decomposition by scales permits us to add progressively the components of the field at smaller and smaller scales, halting the reconstruction when the signal becomes noisy (see [7]).

2.2. Reconstructing other quantities from the radial velocity data

In order to derive the kinematical potential $\Phi(\mathbf{x})$ from the radial velocity field $v_r(\mathbf{x})$, we have to apply on v_r the 'integral along the line-of-sight' operator P (defined in equation 1.1). We then decompose the kinematical potential by scales as done for the velocity in equation (2.1), and the component $\Phi^{(s)}(\mathbf{x})$ is linked to $v_r^{(s)}(\mathbf{x})$ by the wavelet transformed equation (1.1):

$$\Phi^{(s)}(\mathbf{x}) = (Pv_r^{(s)})(\mathbf{x}) = \int_0^1 dl v_r^{(s)}(l\mathbf{x}) \quad (2.7)$$

A practical way to reconstruct the potential is now to introduce the "generalized kernel" $L(s, \mathbf{x}, \mathbf{y})$ (see the illustration figure 1b) by applying the operator P on the reproducing kernel $K(s, \mathbf{x}, \mathbf{y})$:

$$L(s, \mathbf{x}, \mathbf{y}) = P \circ K(s, \mathbf{x}, \mathbf{y}) = \int_0^1 dl K(s, l\mathbf{x}, \mathbf{y}) \quad (2.8)$$

It then follows from equation (2.2) that the component $\Phi^{(s)}(\mathbf{x})$ is directly given by the convolution of the observed radial velocity field $v_r(\mathbf{x})$ by the generalized kernel L which we emphasize depends only on the properties of the wavelet mother function and so is known in principle to arbitrary accuracy, either analytically or numerically. This convolution has the explicit form :

$$\Phi^{(s)}(\mathbf{x}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d^3\mathbf{y} L(s, \mathbf{x}, \mathbf{y}) v_r(\mathbf{y}) \quad (2.9)$$

The important idea is that other quantities of physical significance may be reconstructed in the same way, such as the tangential components of the velocity field, the divergence of the field, the laplacian of the kinematical potential, etc For each such quantity which depends linearly on the velocity field, a specific kernel function similar to L for the potential can be introduced that reduces the reconstruction at a given scale to a convolution with the radial data as in equation (2.9). For example, to the laplacian of the kinematical potential $\nabla^2\Phi$ will be associated a generalized kernel $N(s, \mathbf{x}, \mathbf{y})$ given by :

$$N(s, \mathbf{x}, \mathbf{y}) = \nabla_{\mathbf{x}}^2 L(s, \mathbf{x}, \mathbf{y}) = \int_0^1 dl \nabla_{\mathbf{x}}^2 L(s, l \mathbf{x}, \mathbf{y}) \quad (2.10)$$

This feature avoids in practice the amplification of the errors arising during the successive steps of the reconstruction procedure, especially because quantities of interest are normally extracted from the kinematical potential by the application of differential operators on Φ , a process that is notoriously dangerous when applied to a noisy signal.

3. Our smoothing procedure

3.1. The philosophy

Peculiar radial velocities have been measured for several thousands of galaxies, and these have large statistical uncertainties. But the major difficulty arises from the fact that these galaxies with observed radial velocity are not sampled homogeneously throughout space. The radial velocity field $v_r(\mathbf{x})$ is sampled on a 3-dimensional spatial support (defined by the positions of galaxies) sparsely distributed, with large voids empty of information. One thus has to first smooth the observed radial velocity data before applying a potential reconstruction procedure. This is done in the POTENT method [2,4] by using a tensor window function of large effective diameter 2500 km.s^{-1} compared to the size of the sample (approximately 10000 km.s^{-1}). One thus easily understands the importance of the choice of the smoothing procedure and the attempts to improve upon this procedure (see [9]).

The main weakness we see in the smoothing procedure used by the POTENT method

is that the effective radius of their smoothing window function is everywhere constant throughout the sample. Fictitious information is thus added where voids larger than the effective smoothing radius are present, and information is lost in oversampled regions. Because of the sparseness of the available data, it seems fundamental to avoid this loss of information. We develop below a smoothing scheme based on the use of the wavelet formalism which allows the radius of the smoothing window function to vary from place to place in the sample. In terms of wavelet vocabulary (see section 2.1), we perform the reconstruction of the radial velocity field with a spatially variable cut-off scale $s_c(\mathbf{x})$, halting the wavelet reconstruction at a larger scale in undersampled regions and at smaller scales where the distribution of observed galaxies is dense. Our aim is to extract a smoothed velocity field containing all the information present in the catalog of the observed radial velocity of galaxies but no more.

The resulting map of the cut-off scale $s_c(\mathbf{x})$ is wholly determined by the spatial distribution of the 3-dimensional support (i.e : the positions $\{\mathbf{x}_i\}_{i=1,N}$ of the N galaxies of the catalog). It permits us to compare our output smoothed radial velocity field derived from a known sparsely sampled radial velocity field $\{v_r(\mathbf{x}_i)\}_{i=1,N}$, with the wavelet reconstruction of the true field $v_r(\mathbf{x})$ stopped at the cut-off scale map $s_c(\mathbf{x}) : (W_{s_c(\mathbf{x})}v_r)(\mathbf{x})$. Thus, the errors generated during the successive steps of our smoothing procedure can be quantified.

3.2. The smoothed velocity field

The observed radial velocity field we have to smooth is sampled on a 3-dimensional support inhomogeneously distributed throughout space. We have simulated a typical cosmic radial velocity field and sampled it on the support defined by the real positions (expressed in cartesian supergalactic coordinates) of the galaxies of the MARK II catalog compiled by D. Burstein (416 independent objects are present). Figure 2 shows the simulated radial velocity field on nine cuts passing through a cube of size 10000 km.s^{-1} and centered on our galaxy (herein distances are expressed in km.s^{-1} ; for a Hubble constant H_0 of $100 \text{ km.s}^{-1}/\text{Mpc}$, the size of the sample is 100 Mpc). This non-uniform spatial sampling prevents us in practice from estimating integrals over space such as those of equation (2.2) by their associated discrete riemannian sums over the data positions. However, if we first restore the uniformity to the spatial support, such approximations can afterwards be performed with no prejudice, as long as the spatial extension s of the kernel involved in the convolution of equation (2.2) is greater than the elementary distance between neighbouring data points on the homogeneous support. Our smoothing scheme explores just this possibility.

We call $E_{\mathbf{x}}$ the real 3-dimensional space wherein the spatial support $\{\mathbf{x}_i\}_{i=1,N}$ of the

Figure 2: The simulated radial velocity field $v_r(\mathbf{x}_i)$ at the real positions of the MARK II catalog (the length of arrows is proportional to the amplitude of the field).

Figure 3: The associated field $v'_r(\mu(\mathbf{x}_i))$ in the fictitious E_μ space.

N galaxies of the catalog is inhomogeneously distributed and we define by $\rho(\mathbf{x})$ the spatial distribution of the support in this space. We introduce a mapping μ from this real space $E_{\mathbf{x}}$ into a fictitious 3-dimensional space E_{μ} such that the image $\{\mu_i = \mu(\mathbf{x}_i)\}_{i=1,N}$ of the support under the mapping μ is uniformly distributed in the space E_{μ} :

$$\mu : \begin{cases} E_{\mathbf{x}} \longrightarrow E_{\mu} \\ \mathbf{x} \longmapsto \mu(\mathbf{x}) \end{cases} \quad J_{\mu}(\mathbf{x}) = \left| \det \left[\frac{\partial \mu_j}{\partial x_k}(\mathbf{x}) \right] \right| = \rho(\mathbf{x}) \quad (3.1)$$

In practice, we evaluate the mapping μ using an algorithm (this algorithm is described in the appendix A).

The first step of our smoothing scheme is to associate to the set of data $\{v_r(\mathbf{x}_i)\}_{i=1,N}$ of the real space $E_{\mathbf{x}}$, the set $\{v'_r(\mu_i) = v'_r(\mu(\mathbf{x}_i))\}_{i=1,N}$ in the fictitious E_{μ} space such that :

$$v'_r(\mu(\mathbf{x})) = v_r(\mathbf{x}) \quad (3.2)$$

This operation is illustrated in figure 3 where the $\{v'_r(\mu_i)\}_{i=1,N}$ are shown inside the normalized cube in the fictitious E_{μ} space.

We remark that the function v'_r is now sampled on a uniform support $\{\mu_i = \mu(\mathbf{x}_i)\}_{i=1,N}$ in E_{μ} . It is therefore possible to define an elementary cut-off scale s_{μ} , or minimal resolution length, as the mean distance between 2 neighbouring points in E_{μ} , (herein $s_{\mu} = 0.15$) and to perform in the E_{μ} space the wavelet reconstruction $W_{s_{\mu}}v'_r$ of v'_r halted at the cut-off scale s_{μ} (see equation 2.4). The result of this operation is presented in figure 4. Each components $v_r^{(s)}(\mu)$ involved in the integral over the scales of equation 2.4 is estimated by replacing the spatial convolution of equation 2.2 by its associated discrete riemannian sum over the uniformly distributed set of points $\{\mu_i\}_{i=1,N}$. Since the spatial extension s of the kernel K involved in the convolution is indeed always greater than the separation between the μ_i 's, this standard estimation operation can be done without creating fictitious information. Note that $W_{s_{\mu}}v'_r(\mu)$ is now defined for every μ of E_{μ} and contains all of the information that can be extracted from the $\{v'_r(\mu_i)\}_{i=1,N}$ data in the fictitious E_{μ} space.

The last step of our smoothing procedure is to return to the real space $E_{\mathbf{x}}$ through the inverse mapping μ^{-1} . Our output smoothed velocity field $(Mv_r)(\mathbf{x})$ is finally the field corresponding to $W_{s_{\mu}}v'_r$ in the real space $E_{\mathbf{x}}$ namely :

$$(Mv_r)(\mathbf{x}) = (W_{s_{\mu}}v'_r)(\mu(\mathbf{x})) \quad (3.3)$$

This operation is illustrated figure 5. Since the mapping μ establishes a one-to-one correspondance between points of the real space $E_{\mathbf{x}}$ and those of the fictitious E_{μ} , our output smoothed radial velocity field $(Mv_r)(\mathbf{x})$ contains the maximum of information

Figure 4: The wavelet reconstruction $(W_{s_\mu} v'_r)(\mu)$ of v'_r halted at the cut-off scale s_μ .

which can be extracted from the $\{v'_r(\mu_i)\}_{i=1,N}$ and thus from the observed radial velocities $\{v_r(\mathbf{x}_i)\}_{i=1,N}$ of the N galaxies in the catalogue. In this way our smoothing procedure is minimal (no loss of information).

3.3. Correspondence with a theoretical quantity

Thanks to the scale resolution properties of the wavelet transform, our output smoothed radial velocity field $(Mv_r)(\mathbf{x})$ can be expressed as a theoretical quantity. We prove in appendix B that, so long as the mapping μ satisfies a validity condition (see below equation 3.5), the following equality holds :

$$(Mv_r)(\mathbf{x}) = (W_{s_c(\mathbf{x})} v_r)(\mathbf{x}) \quad \text{with} \quad s_c(\mathbf{x}) = \frac{s_\mu}{\rho(\mathbf{x})^{1/3}} \quad (3.4)$$

Thus our output smoothed radial velocity field may be identified with the wavelet reconstruction of v_r halted at the inverse map of the cut-off scale $s_c(\mathbf{x})$ which varies with \mathbf{x} . We show in figure 6 the cut-off scale inverse map corresponding to the spatial distribution previously presented in figure 2. The value of $s_c(\mathbf{x})$ is derived from the jacobian associated with the mapping μ (see appendix A). The lower the density at the position \mathbf{x} , the larger is the corresponding cut-off scale.

We exhibit in figure 7 the wavelet reconstruction of the previously simulated radial velocity field halted at the cut-off scale $s_c(\mathbf{x})$. We notice that even if the main features

Figure 5: Our output smoothed radial velocity field $(Mv_r)(\mathbf{x})$ (in the real space $E_{\mathbf{x}}$).

remain, our smoothed radial velocity field $(Mv_r)(\mathbf{x})$ of figure 5 differs in detail from $(W_{s_c(\mathbf{x})}v_r)(\mathbf{x})$. The reason for this discrepancy is that the mapping μ doesn't in fact satisfy the validity condition (appendix B) which stipulates that for every \mathbf{x} and vector \mathbf{h} :

$$\text{if } \|\mathbf{h}\| \leq s_c(\mathbf{x}), \quad \left\| \left[\frac{\partial \mu_j}{\partial x_k}(\mathbf{x}) \right] \cdot [\mathbf{h}] \right\| \approx \left| \det \left[\frac{\partial \mu_j}{\partial x_k}(\mathbf{x}) \right] \right|^{1/3} \times \|\mathbf{h}\| \quad (3.5)$$

or in other words that the mapping μ is locally equivalent to a rotation-dilation transformation (see appendix B).

However, this discrepancy between our output smoothed velocity field $(Mv_r)(\mathbf{x})$ and the wavelet reconstruction $(W_{s_c(\mathbf{x})}v_r)(\mathbf{x})$ of the true radial velocity field $v_r(\mathbf{x})$ halted at the cut-off scale map $s_c(\mathbf{x})$ can be quantified by analysing the spatial distribution of the N galaxies of the sample. Thus at each position \mathbf{x} in the sampled volume, we can define a "relative sampling error" $E(\mathbf{x})$ computed as follows :

$$E^2(\mathbf{x}) = \frac{\sum_{i=1}^N [K(s_\mu, \mu(\mathbf{x}), \mu(\mathbf{y}_i)) - 1/\rho(\mathbf{y}_i) K(s_c(\mathbf{x}), \mathbf{x}, \mathbf{y}_i)]^2}{\sum_{i=1}^N [K(s_\mu, \mu(\mathbf{x}), \mu(\mathbf{y}_i))]^2} \quad (3.6)$$

This 'relative sampling error map' $E(\mathbf{x})$ gives at each position \mathbf{x} the relative error between $(Mv_r)(\mathbf{x})$ and $(W_{s_c(\mathbf{x})}v_r)(\mathbf{x})$ as a fraction of the amplitude of the true radial velocity field $v_r(\mathbf{x})$. The relative sampling error map $E(\mathbf{x})$ for the spatial distribution of the galaxies

Figure 6: Isocontours of the cut-off scale map $s_c(\mathbf{x})$ (Normal, heavy and dotted contours are resp., 750, 1500 and 2250 $km.s^{-1}$).

Figure 7: The wavelet reconstruction $(W_{s_c(\mathbf{x})}v_r)(\mathbf{x})$ of the simulated radial velocity field $v_r(\mathbf{x})$ halted at the cut-off scale map $s_c(\mathbf{x})$.

Figure 8: Isocontours of the relative sampling errors map $E(\mathbf{x})$ (Normal, heavy and dotted contours are resp., 15%, 25% and 50%).

of the MARK II catalogue is shown in figure 8. In practice we can discard the regions of the sample where the sampling errors are too high.

4. The kinematical potential $\Phi(\mathbf{x}) = (Pv_r)(\mathbf{x})$

We have shown in section 3.3 that, in the regions of the sampled volume where the validity condition is verified (i.e : the domains where the relative sampling errors $E(\mathbf{x})$ are low), our smoothed radial velocity field $(Mv_r)(\mathbf{x})$ can be identified with $(W_{s_c(\mathbf{x})}v_r)(\mathbf{x})$. As we suggested in the introduction, there are good reasons to believe that the cosmic velocity field is irrotational up to an ill-defined scale and thus that a kinematic potential $\Phi(\mathbf{x})$ can be inferred by integrating the cosmic radial velocity field $v_r(\mathbf{x})$ along the line-of-sight (equation 1.1). It turns out that the ill-defined scale of irrotationality is generally presumed to be smaller than the cut-off scales $s_c(\mathbf{x})$ which limit the spatial resolution of the accessible cosmic radial velocity field $(W_{s_c(\mathbf{x})}v_r)(\mathbf{x})$. It becomes thus possible to extract the velocity potential $\Phi(\mathbf{x})$ by applying the ‘integral-along-the-line-of-sight’ operator P either to $(W_{s_c(\mathbf{x})}v_r)(\mathbf{x})$ or to the smoothed radial velocity field $(Mv_r)(\mathbf{x})$.

Unfortunately, this operator P has a non-local character. Hence, the potential derived from the smoothed radial velocity field $P \circ W_{s_c(\mathbf{x})}v_r$ differs from the smoothed potential of the velocity field $W_{s_c(\mathbf{x})} \circ Pv_r$ (and of course from the non-smoothed kinematical potential

Figure 9: Isocontours of the potential of the smoothed radial velocity field $(P \circ W_{s_c(\mathbf{x})} v_r)(\mathbf{x})$ (the contour spacing is $2.5 \cdot 10^5 (km.s^{-1})^2$, negative contours are dotted, heavy contour is 0).

Figure 10: Isocontours of the smoothed simulated potential $(W_{s_c(\mathbf{x})} \circ P v_r)(\mathbf{x})$ (the contour spacing is $2.5 \cdot 10^5 (km.s^{-1})^2$, negative contours are dotted, heavy contour is 0).

$\Phi = Pv_r$) (see appendix C). We illustrate this discrepancy by plotting in figure 9 the potential derived from $(W_{sc}(\mathbf{x})v_r)(\mathbf{x})$ and in figure 10 the smoothed simulated potential $(W_{sc}(\mathbf{x})\Phi)(\mathbf{x})$. We want to emphasize that this behaviour is not due to the way we smooth the radial velocity field but is intrinsically linked to the nature of the operator P .

The point has its importance since the kinematical potential (or the reconstructed 3-dimensional velocity field) derived from catalogues of radial peculiar velocities is often considered as reference data for other studies (for example Saunders *et al.* (1991), Dekel *et al.* (1993)). As an example note that, above the scale of irrotationality, the total mass density perturbation field $\delta_t(\mathbf{x})$ is linked to the kinematical potential $\Phi(\mathbf{x})$ through the Poisson equation ($\delta_t(\mathbf{x}) \propto \nabla^2\Phi(\mathbf{x})$). Adopting the hypothesis that the luminous matter traces the mass ($\delta_t(\mathbf{x}) \propto \delta_l(\mathbf{x})$), a smoothed total mass density field may be extracted from observations of the spatial distribution of galaxies. Unfortunately for the intercomparison of such independent results :

$$(W_{sc}\delta_t)(\mathbf{x}) = (\nabla^2(W_{sc} \circ Pv_r))(\mathbf{x}) \neq (\nabla^2(P \circ W_{sc}v_r))(\mathbf{x}) \quad (4.1)$$

One thus should be very cautious when comparing the density or kinematical potential derived from observed radial peculiar velocity catalogues with similar quantities obtained from other studies, such as those based on number galaxy counts.

5. Conclusions

We have presented a method, based on the properties of the wavelet transforms, for smoothing a field sampled on a support inhomogeneously distributed throughout the space of interest. Our smoothing scheme is minimal (no loss of information) and our output smoothed field can be identified with a well-defined theoretical quantity, as long as the spatial support of the field satisfies certain criteria. We expect this technique to be of quite general importance. The particular application of this smoothing scheme to the observed cosmic radial velocity field discussed above, reveals some apparently previously unrecognized limitations concerning the reconstruction of the kinematic potential from the smoothed radial velocity field. Indeed, we prove that this potential will generally differ a priori from a similar quantity obtained from other cosmological studies. The existence of this kind of error now has to be taken into account.

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Appendix A : The μ mapping

In practice, we evaluate the mapping μ using the following algorithm. We first choose a direction (the x_1 -axis for example). Along this direction, we sort the N points of the sample by the increasing value of their x_1 -coordinate. To the object of rank i , we then affect the value $\mu_1 = i/N$ to its μ_1 -coordinate, the first component of the mapping μ . This procedure ensures us that the set of the μ_1 -coordinates of the sampled objects are uniformly distributed along the μ_1 -axis in the fictitious space E_μ . We now divide the cube in slices perpendicular to the x_1 -axis (7 x_1 -slices for this sample) such that each slice contains approximatively the same number N_1 of data points. The non-uniform sampling in the E_x space implies that each slice have its own width. Nevertheless, the widths of these slices become equal along the μ_1 -axis in the space E_μ .

The next step of our algorithm is to repeat this construction for each x_1 -slice, choosing the x_2 -axis as the sorting direction. Inside each x_1 -slice, the object of rank j has a $\mu_2 = j/N_1$. We thus obtain μ_2 -coordinates uniformly distributed inside each x_1 -slice, and so for the set of all slices in the fictitious E_μ space. The x_1 -slices are afterwards divided again in slices perpendicular to the x_2 -axis (7 for this sample) containing the same number N_2 of objects and each x_1x_2 -slice is sorted along the x_3 -axis (for an object of rank k , $\mu_3 = k/N_2$). N_2 is 8 or 9 in our case, depending on the x_1x_2 -slice treated.

At the output of this algorithm, we then obtain a set of N points $\{\mu_i = \mu(\mathbf{x}_i)\}_{i=1,N}$ uniformly distributed inside a normalized cube in the fictitious space E_μ . A continuous version of the μ mapping as well as its inverse mapping μ^{-1} can be computed, if necessary, by using a linear interpolation on the μ_1 , μ_2 and μ_3 components of the transformation μ .

In order to evaluate quantities such as the cut-off scale map $s_c(\mathbf{x})$ (section 3.3.), the spatial density distribution of the support $\rho(\mathbf{x})$ in the real space E_x has to be evaluated. We then first estimate at the position \mathbf{x} the 9 terms $(\partial\mu_j/\partial x_k)(\mathbf{x})$ (j and k vary from 1 to 3) of the transformation coordinate matrix associated with the mapping μ . We then derive from this matrix its jacobian $J_\mu(\mathbf{x})$ and finally adopt for the density distribution $\rho(\mathbf{x})$ a smoothed version of $J_\mu(\mathbf{x})$.

Appendix B : Comparison between $(Mv_r)(\mathbf{x})$ and $(W_{s_c(\mathbf{x})}v_r)(\mathbf{x})$

Our goal here is to identify our output smoothed radial velocity field $(Mv_r)(\mathbf{x})$ with a theoretical quantity derived from the true radial velocity field $v_r(\mathbf{x})$. Our smoothing procedure ensures us that the following equality holds (see equation 3.3) :

$$(Mv_r)(\mathbf{x}) = (W_{s_\mu}v'_r)(\mu(\mathbf{x})) \quad (B.1)$$

In the fictitious E_μ space, $(W_{s_\mu}v'_r)(\mu)$ is the wavelet reconstruction of the field $v_r(\mu)$ halted

at the cut-off scale s_μ . It satisfies (see equations 2.3, 2.4 and 2.5) :

$$v'_r(\mu) = W v'_r(\mu) = W_{s_\mu} v'_r(\mu) + W^{s_\mu} v'_r(\mu) \quad (B.2)$$

Now, Let us focus on the measurable component $(W^{s_\mu} v'_r)(\mu(\mathbf{x}))$. In the fictitious E_μ space, it can be expressed as follows (see equations 2.2 and 2.5) :

$$(W^{s_\mu} v'_r)(\mu(\mathbf{x})) = \int_0^{s_\mu} \frac{ds}{s} \int_{E_\mu} d^3\tau v'_r(\tau) K(s, \mu(\mathbf{x}), \tau) \quad (B.3)$$

Recalling equation 3.2 and introducing \mathbf{y} such that $\tau = \mu(\mathbf{y})$, this equation becomes in the real space $E_{\mathbf{x}}$:

$$(W^{s_\mu} v'_r)(\mu(\mathbf{x})) = \int_0^{s_\mu} \frac{ds}{s} \int_{E_{\mathbf{x}}} \rho(\mathbf{y}) d^3\mathbf{y} v_r(\mathbf{y}) K(s, \mu(\mathbf{x}), \mu(\mathbf{y})) \quad (B.4)$$

At this stage, we have to use the specific properties of the reproducing kernel K . This kernel K can be derived from an "analysing" function k of unit spatial extension due to the following symetry (see [7]) :

$$K(s, \mathbf{x}, \mathbf{y}) = \frac{1}{s^3} k\left(\frac{\|\mathbf{x} - \mathbf{y}\|}{s}\right) \quad (B.5)$$

Hence, for every real number α , the kernel satisfies :

$$K(s, \alpha\mathbf{x}, \alpha\mathbf{y}) = \frac{1}{\alpha^3} K(s/\alpha, \mathbf{x}, \mathbf{y}) \quad (B.6)$$

Keeping this in mind, we develop the mapping μ to the first order of its vectorial taylor series with respect to \mathbf{x} or \mathbf{y} . The difference $\mu(\mathbf{x}) - \mu(\mathbf{y})$ involved in the kernel K intervening in equation B.4 can then be approximated :

$$\|\mu(\mathbf{x}) - \mu(\mathbf{y})\| \approx \left\| \left[\frac{\partial \mu_j}{\partial x_k}(\mathbf{x}) \right] \cdot [\mathbf{x} - \mathbf{y}] \right\| \approx \left\| \left[\frac{\partial \mu_j}{\partial x_k}(\mathbf{y}) \right] \cdot [\mathbf{x} - \mathbf{y}] \right\| \quad (B.7)$$

This approximation is in practice performed inside regions of the E_μ space not larger than the cut-off scale s_μ (these regions are in fact defined by the spatial extension of the kernel K involved in the spatial convolution of the equation B.4).

The next step is to make the crucial assumption that the mapping μ can be locally identified with a rotation-dilation transformation, or in other words that for $\|\mathbf{h}\| \leq s_\mu/\rho^{1/3}(\mathbf{x})$:

$$\left\| \left[\frac{\partial \mu_j}{\partial x_k}(\mathbf{x}) \right] \cdot [\mathbf{h}] \right\| \approx \left| \det \left[\frac{\partial \mu_j}{\partial x_k}(\mathbf{x}) \right] \right|^{1/3} \times \|\mathbf{h}\| = \rho(\mathbf{x})^{1/3} \times \|\mathbf{h}\| \quad (B.8)$$

This equation is certainly not valid for every point \mathbf{x} of the sampled volume. Nevertheless, in the regions of space where this validity condition does hold, the following approximation is valid (see equations B.5, B.6, B.7 and B.8) :

$$\text{if } s \leq s_\mu, \quad K(s, \mu(\mathbf{x}), \mu(\mathbf{y})) \approx \frac{1}{\rho(\mathbf{y})} K(s/\rho(\mathbf{x})^{1/3}, \mathbf{x}, \mathbf{y}) \quad (B.9)$$

Then we can define a cut-off scale map $s_c(\mathbf{x})$ in the $E_{\mathbf{x}}$ space :

$$s_c(\mathbf{x}) = \frac{s_\mu}{\rho(\mathbf{x})^{1/3}} \quad (B.10)$$

and it turns out that if we introduce S such that $S = s/\rho(\mathbf{x})^{1/3}$, equations B.4, B.9 and B.10 give :

$$(W^{s_\mu} v'_r)(\mu(\mathbf{x})) \approx \int_0^{s_c(\mathbf{x})} \frac{dS}{S} \int_{E_{\mathbf{x}}} d^3\mathbf{y} v_r(\mathbf{y}) K(S, \mathbf{x}, \mathbf{y}) = (W^{s_c(\mathbf{x})} v_r)(\mathbf{x}) \quad (B.11)$$

Finally, because of equations 2.3, B.1, B.2 and B.11, the following equality holds in the regions of the sampled volume where the validity condition of equation B.8 is satisfied :

$$(Mv_r)(\mathbf{x}) = (W_{s_c(\mathbf{x})} v_r)(\mathbf{x}) \quad (B.12)$$

Thus, our output smoothed radial velocity field $(Mv_r)(\mathbf{x})$ can be compared to the wavelet reconstruction $(W_{s_c(\mathbf{x})} v_r)(\mathbf{x})$ of the real radial velocity field $v_r(\mathbf{x})$ halted at the cut-off scale map $s_c(\mathbf{x})$. In practice we evaluate the "degree of approximation" of the validity condition by comparing the 2 sides of the equation B.9 integrated over the space at the fixed scale s_μ . This quantity $E(\mathbf{x})$ is defined as follows :

$$E^2(\mathbf{x}) = \frac{\sum_{i=1}^N [K(s_\mu, \mu(\mathbf{x}), \mu(\mathbf{y}_i)) - 1/\rho(\mathbf{y}_i) K(s_c(\mathbf{x}), \mathbf{x}, \mathbf{y}_i)]^2}{\sum_{i=1}^N [K(s_\mu, \mu(\mathbf{x}), \mu(\mathbf{y}_i))]^2} \quad (B.13)$$

This relative sampling errors map $E(\mathbf{x})$ gives at each position \mathbf{x} the relative error between $(Mv_r)(\mathbf{x})$ and $(W_{s_c(\mathbf{x})} v_r)(\mathbf{x})$ with respect to the amplitude of the true radial velocity field $v_r(\mathbf{x})$.

Appendix C : $P \circ W_{s_c(\mathbf{x})} v_r \neq W_{s_c(\mathbf{x})} \circ P v_r$

We prove in this appendix that the 'integral-along-the-line-of-sight operator' P doesn't commute with the necessary smoothing operation that must first be performed on the radial velocity field $v_r(\mathbf{x})$. This behaviour doesn't depend on the particular nature of our preliminary smoothing, rather it remains true more generally even if for example the effective radius of the smoothing window function is constant throughout the sampled

volume as in POTENT.

Let us express the potential derived from the smoothed radial velocity field $P \circ W_{s_c} v_r$ in terms of the smoothing window function introduced equation 2.6 (herein , s_c is chosen constant throughout the space since otherwise our argument is even stronger) :

$$(P \circ W_{s_c} v_r)(\mathbf{x}) = \int_0^1 dl \int_{E_{\mathbf{x}}} d^3 \mathbf{y} v_r(\mathbf{y}) g(s_c, l\mathbf{x}, \mathbf{y}) \quad (C.1)$$

$(P \circ W_{s_c} v_r)(\mathbf{x})$ is the integral along the line-of-sight of the field v_r smoothed with a window function of constant effective radius s_c from \mathbf{O} to \mathbf{x} .

In the same way, let us express the smoothed version at the scale s_c of the potential $(W_{s_c} \circ P v_r)(\mathbf{x})$:

$$(W_{s_c} \circ P v_r)(\mathbf{x}) = \int_{E_{\mathbf{x}}} d^3 \mathbf{y} \int_0^1 dl v_r(l\mathbf{y}) g(s_c, \mathbf{x}, \mathbf{y}) \quad (C.2)$$

Because of the specific properties of the wavelet smoothing window function, it turns out that (see equation 2.6, B.5 and B.6) :

$$g(s_c, \mathbf{x}, \mathbf{y}) = l^3 g(ls_c, l\mathbf{x}, l\mathbf{y}) \quad (C.3)$$

Hence equation C.2 can be re-expressed, if we introduce \mathbf{Y} such that $\mathbf{Y} = l\mathbf{y}$:

$$(W_{s_c} \circ P v_r)(\mathbf{x}) = \int_0^1 dl \int_{E_{\mathbf{x}}} d^3 \mathbf{Y} v_r(\mathbf{Y}) g(ls_c, l\mathbf{x}, \mathbf{Y}) \quad (C.4)$$

$(W_{s_c} \circ P v_r)(\mathbf{x})$ is then the integral along the line-of-sight of the field v_r smoothed with a window function with a variable effective radius ls_c all along the line-of-sight, from 0 at the observator position \mathbf{O} to s_c in \mathbf{x} . It thus follows that :

$$P \circ W_{s_c} v_r \neq W_{s_c} \circ P v_r \quad (C.5)$$

as was required.

References

- [1] Bertschinger E. and Dekel A. (1989) *Astrophys. J. (Letters)*, **336**, L5.
- [2] Bertschinger E., Dekel A., Faber S.M. Dressler A. and Burstein D. (1990) *Astrophys. J.*, **364**, 370.
- [3] Daubechies I. (1992) in *Ten lectures on Wavelets*, CBMS-NSF Regional Conference Series in Applied Mathematics, SIAM.
- [4] Dekel A., Bertschinger E. and Faber S.M. (1990) *Astrophys. J.*, **364**, 349.
- [5] Dekel A., Bertschinger E., Yahil A., Strauss M., Davis M. and Huchra J. (1993)

Astrophys. J., **412**, 1.

[6] Grossmann A., Kronland-Martinet R. and Morlet J. (1990), in *Wavelets*, Proc. of the International Conference, Tchamitchian ed. , Springer-Verlag.

[7] Rauzy S., Lachièze-Rey M. and Henriksen R.N. (1993) *Astr. Astrophys.*, **273**, 357.

[8] Saunders W., Rowan-Robinson M., Lawrence A., Crawford J., Ellis R., Frenk C.S., Parry I., Xiaoyang X., Allington-Smith J., Estathiou G. and Kaiser N. (1991) *Nature*, **349**, 32.

[9] Simmons J.F.L., Newsam A. and Hendry M.A. (1994) *Astr. Astrophys.*, in press.